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METELITSYN'S THEOREMS†

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Metelitsyn [1, 2] published several theorems on the stability of linear systems of the form

$$A\ddot{q}+(B+G)\dot{q}+(C+\mathscr{E})q=0 \tag{1}$$

where q is the n-dimensional vector of generalized coordinates, A is the symmetrical and positive-definite matrix of the kinetic energy, B is the symmetric matrix of dissipative forces, G is the skew-symmetric matrix of gyroscopic forces, C is the symmetric matrix of potential forces and $\mathscr{E}$ is the skew-symmetric matrix of strictly non-conservative forces.

As regards such systems, the question has been raised as to the possibility of judging their stability directly from the structure of the forces applied, omitting a calculation of the characteristic exponents. Thus, the question was first raised by Thomson and Tait [3], but no attention was paid by them to systems with strictly non-conservative forces defined by the matrix $\mathscr{E}$.

Allowance for these forces is interesting for two reasons. First, these forces are very common in nature and in engineering; it is sufficient, for example, to recall that such a threatening phenomenon as flutter is induced by these very forces. Second, when allowance is made for these forces, the problem acquires known completion, since in this case linear systems with constant coefficients (1) have the maximum general form.

Metelitsyn's results provoked contradictory responses, and occasionally doubts were expressed as to their correctness. This is due to two factors: the use of unconventional terminology, and a peculiar, often excessively economic manner of presentation, which required additional explanations of his position by the author.

Metelitsyn's theorems, however, are a very useful tool for investigating systems at the preliminary stage of analysing their stability, on account of which they are used in practice.

All this prompts a discussion of these theorems in order to understand the degree to which the doubts that occasionally arise are justified.

To begin with, following Metelitsyn, we will derive, in slightly greater detail, the main inequality subsequently used in proving the theorems.

The solution of Eq. (1) is sought in the form

$$q = he^{\mu t} \tag{2}$$

which leads to the following algebraic system

$$[A\mu^2 + (B+G)\mu + C + \mathscr{E}]h = 0 \tag{3}$$

We will determine an implicit function $\mu(h)$, where $\mu \in \mathbb{C}$ and $h \in \mathbb{C}^n$, in the following way. We multiply equality (3) by the Hermitian conjugate vector h^*, as a result of which we obtain

$$T\mu^2 + (D+i\Gamma)\mu + V + iE = 0 \tag{4}$$
$$T = h^*Ah, \quad D = h^*Bh, \quad i\Gamma = h^*Gh, \quad V = h^*Ch, \quad iE = h^*\mathscr{E}h$$

The variables T, D, Γ, V and E are real functions of h.

Equation (4) can be solved for μ. We have

$$\mu = (-D - i\Gamma \pm \sqrt{(D+i\Gamma)^2 - 4T(V+iE)})/(2T) \tag{5}$$

i.e. an explicit function $\mu(h)$ has been obtained with the feature that, if the vector h satisfies system (3), then μ reverts to the characteristic exponent corresponding to this vector for one of the signs in front of the root. It can also be noted that the function $\mu(h)$ possesses extremal features: its partial derivatives with respect to the components of the vector h vanish on solutions of system (3). However, the latter feature is not used subsequently.

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In any case, if conditions are imposed on the functions $T(h)$, $D(h)$, $\Gamma(h)$ and $E(h)$ so that $\operatorname{Re}\mu < 0$ for any complex vector h, then this is sufficient for this function and the eigenvectors of system (3) to lie in the left-hand half-plane. In other words, this is sufficient for all the characteristic exponents of system (1) to lie in the left-hand half-plane.

We will obtain this condition. To do this, we first separate the real part of the root in the numerator of expression (5). We have

$$\sqrt{(D+i\Gamma)^2-4T(V+iE)}=\sqrt{\Delta+i(2D\Gamma-4TE)}=\sqrt{r(\cos\varphi+i\sin\varphi)}=\sqrt{r}\left(\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}\right)$$

$$\Delta=D^2-\Gamma^2-4TV,\quad r^2=\Delta^2+(2D\Gamma-4TE)^2,\quad \cos\frac{\varphi}{2}=\sqrt{\frac{1+\cos\varphi}{2}}=\sqrt{\frac{r+\Delta}{2r}}$$

Hence, the real part of the characteristic exponent is obtained as

$$\operatorname{Re}\mu=(-D\pm\sqrt{(r+\Delta)/2})/(2T)$$

By virtue of the fact that the quadratic form of the kinetic energy is positive-definite, the condition for both roots of Eq. (3) to be negative thus reduces to the inequality

$$-D\pm\sqrt{(r+\Delta)/2}<0$$

In this inequality it is possible to eliminate the roof if it is assumed that $D \geqslant 0$. We then obtain

$$D^2+\Gamma^2+4TV>r \tag{6}$$

Note that condition (6) guarantees that not only the true root of the characteristic equation of system (1) but also the superfluous second root of Eq. (3) lie in the left-hand half-plane. Thereby, the condition of stability obtained below may be solely sufficient.

Squaring both sides of inequality (6), we obtain

$$TE^2-\Gamma DE<D^2V \tag{7}$$

We now turn our attention to the fact that the case when $D = 0$ is admissible in all discussions based on inequality (7). This does not contradict Metelitsyn's assumption that the quadratic form of the dissipative forces is positive-definite, since this assumption is not a feature of the system but a feature of the forces that can be applied but may not be applied. This is the essence of Thomson and Tait's idea of the structural conditions of stability, unlike the algebraic conditions.

Inequality (7), which is a strict, effectively verified, sufficient condition for the asymptotic stability of linear systems with constant coefficients of maximum general form, is also the main result of Metelitsyn's papers. Examples of arguments that can be presented on the basis of this inequality are given by Metelitsyn in the form of theorems. Other examples of this kind can be given.

Before analysing the theorems, we point out that stability in them is understood to mean asymptotic stability. This is directly mentioned in the very first phrase of the corresponding paper in [2]. In those cases where it is a matter of simple stability, Metelitsyn uses the terms "static" or "permanent" stability. These terms are not used now but they are tied to a context in which their sense is clear and give rise to no misunderstandings. Finally, note that Metelitsyn believed inequality (7) to be *necessary and sufficient*, since he assumed that both roots given by formula (5) belong to the system. However, he provided no proof of this.

For this reason, those theorems in which inequality (7) is regarded as necessary cannot be considered proved. This applies to Theorems 1 and 7. In Theorem 3 it is necessary to remove the word "only". The correctness of Theorems 2, 3 and 4 readily follows from inequality (7). Here, it is necessary to bear in mind that a phrase of the type "can be made stable by adding such-and-such forces" should not be understood to mean that stability is guaranteed by applying any forces of this kind. It is claimed that the necessary forces of the structure indicated will be found. For this reason, examples of instability when specific forces of this kind are applied are not contrary examples.

Theorems 5 and 6 do not follow from the inequality established. They stem from formula (5) and permit an obvious addition: if the rank of the matrix of gyroscopic forces is lower than the dimensionality of the system, then, besides the roots indicated in the theorem, roots appear that tend to finite non-zero limits with the norm of this matrix tending to infinity. A comparison of matrices with respect to norms is also able to interpret the meaning of the term "predominate" which occurs in Theorem 6.

To sum up, it can be concluded that Metelitsyn's main result is the extremely interesting inequality (7). His theorems are an illustration of how this inequality can be used to analyse stability.

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